(5) The Canonical Commutation Relations

$$[\tilde{x}_i, \tilde{x}_j] = 0, \quad [\tilde{p}_i, \tilde{p}_j] = 0, \quad [\tilde{x}_i, \tilde{p}_j] = i\hbar\,\delta_{ij}$$

other useful identities.

c-number

- $[A, A] = 0, \quad [A, B] = -[B, A], \quad [A, c] = 0.$
- $[A+B, C] = [A, C] + [B, C]$
- $[A, BC] = [A, B]C + B[A, C]$
- $[A, [B, C]] + [B, [C, A]] + [C, [A, B]] = 0$

(Jacobi identity).

1.7 Wave functions in position and momentum space.

(1) Position-Space wave function

→ Base kets = "position" kets : $\tilde{x}|x\rangle = x|x\rangle$

orthogonality : $\langle x|x'\rangle = \delta(x-x')$

|| completeness rel.

$\int dx\, |x\rangle\langle x| = 1$

→ Wave function

a physical state $|\alpha\rangle = \int dx\, |x\rangle\langle x|\alpha\rangle$

$= \int dx\, \psi_\alpha(x)\, |x\rangle.$

≈ expansion coefficient of x-ket "localized" at x.

- wave function in position space

$$\boxed{\psi_\alpha(x) = \langle x|\alpha\rangle.}$$

|| probability for the particle to be found in $[x, x+dx]$

$= |\psi_\alpha|^2 dx$

- Inner product

$$\langle\beta|\alpha\rangle = \int dx\, \langle\beta|x\rangle\langle x|\alpha\rangle$$

$$= \int dx\, \psi_\beta^*(x)\, \psi_\alpha(x).$$

- Change of base kets

$$|\alpha\rangle = \sum_a |a\rangle\langle a|\alpha\rangle$$ ... What about the wave functions?

↳ $$\langle x|\alpha\rangle = \sum_a \langle x|a\rangle \langle a|\alpha\rangle$$

→ $$\psi_\alpha(x) = \sum_a c_a\, u_a(x)$$

For example, choosing $|a\rangle$ to be the eigenket of Hamiltonian,

then. $u_\alpha(x)$ is the eigenfunction $(\langle x|\alpha\rangle)$.

and $\psi_\alpha(x)$ is a general form of wave function.

(a particle in a box : $u_1(x) = e^{ikx}$, $u_2 = e^{-ikx}$

$\psi(x) = c_1 e^{ikx} + c_2 e^{-ikx}$.

- matrix element $\langle\beta|A|\alpha\rangle$ of an operator A.

$$\langle\beta|A|\alpha\rangle = \int dx' \int dx'' \langle\beta|x'\rangle\langle x'|A|x''\rangle\langle x''|\alpha\rangle$$

$$= \int dx' \int dx'' \psi_\beta^*(x') \langle x'|A|x''\rangle \psi_\alpha(x'')$$

ex. $A = \tilde{x}^2$

$$\langle x'|\tilde{x}^2|x''\rangle = x'^2\langle x'|x''\rangle = x'^2\delta(x'-x'').$$

$$\langle\beta|x^2|\alpha\rangle = \int dx' \int dx'' \psi_\beta^*(x')\, x'^2\, \delta(x'-x'')\, \psi_\alpha(x'')$$

do it

$$= \int dx' \psi_\beta^*(x')\, x'^2\, \psi_\alpha(x')$$

In general, $$\langle\beta|f(\tilde{x})|\alpha\rangle = \int dx\, \psi_\beta^*(x)\, f(x)\, \psi_\alpha(x)$$

(2) Momentum operator in the position basis

Let's play with the infinitesimal translation operator.

$$\left(1 - \frac{i\tilde{p}\,\delta x}{\hbar}\right)|\alpha\rangle = \int dx'\, J(\delta x)\,|x'\rangle\langle x'|\alpha\rangle$$

$$= \int dx'\, |x'+\delta x\rangle\langle x'|\alpha\rangle \qquad \& \quad x' \to x'-\delta x$$

* $f(x) \equiv \langle x|\alpha\rangle$

$$= \int dx'\, |x'\rangle\langle x'-\delta x|\alpha\rangle$$

↓ Taylor series expansion

$f(x) \simeq f(x') + \frac{\partial}{\partial x} f(x)\Big|_{x'}(x-x') + \cdots$

$f(x'-\delta x) = f(x') - \delta x \cdot \frac{\partial}{\partial x} f(x)\Big|_{x'} + \cdots$

$$= \int dx'\, |x'\rangle\left[\langle x'|\alpha\rangle - \delta x \frac{\partial}{\partial x'}\langle x'|\alpha\rangle + \cancel{O}(\delta x^2)\right]$$

$$\Rightarrow \cancel{|\alpha\rangle} - \frac{i\delta x}{\hbar}\tilde{p}|\alpha\rangle = \cancel{\int dx'\,|x'\rangle\langle x'|\alpha\rangle} - \delta x \int dx'\, |x'\rangle \frac{\partial}{\partial x'}\langle x'|\alpha\rangle$$

↕ compare!

$$\Rightarrow \tilde{p}|\alpha\rangle = \int dx' \int dx''\, |x'\rangle\langle x'|\tilde{p}|x''\rangle\langle x''|\alpha\rangle$$

$$\therefore \quad \boxed{\langle x'|\tilde{p}|x''\rangle = -i\hbar \frac{\partial}{\partial x'} \cdot \delta(x'-x'')}$$

☆☆☆

- For general kets $|\alpha\rangle$ and $|\beta\rangle$,

$$\langle\beta|\tilde{p}|\alpha\rangle = \int dx' \int dx''\, \langle\beta|x'\rangle\langle x'|\tilde{p}|x''\rangle\langle x''|\alpha\rangle$$

$$= \int dx' \int dx''\, \psi^*_\beta(x') \cdot \left(-i\hbar \frac{\partial}{\partial x'}\delta(x'-x'')\right)\cdot \psi_\alpha(x'')$$

↑ integrate first.

$$= \int dx'\, \psi^*_\beta(x')\left(-i\hbar\frac{\partial}{\partial x'}\right)\psi_\alpha(x')$$

- For $\tilde{p}^n$,

$$\langle x'|\tilde{p}^n|\alpha\rangle = \left(-i\hbar \cdot \frac{\partial}{\partial x'}\right)^n \langle x'|\alpha\rangle$$

$$\langle\beta|\tilde{p}^n|\alpha\rangle = \int dx'\, \psi^*_\beta(x')\left[-i\hbar\frac{\partial}{\partial x'}\right]^n \psi_\alpha(x')$$

(3) Momentum-Space Wave ~~function~~.

Similarly for the base eigenkets in the p-basis,

$$\tilde{p}|p\rangle = p|p\rangle, \qquad \langle p'|p''\rangle = \delta(p'-p'')$$

completeness relation $\qquad 1 = \int dp\, |p\rangle\langle p|$,,

- momentum-space wave function.

$$\langle p|\alpha\rangle = \phi_\alpha(p) \quad \text{as} \quad |\alpha\rangle = \int dp\, |p\rangle \underbrace{\langle p|\alpha\rangle}_{\phi_\alpha(p)}$$

- transformation : $x \longleftrightarrow p$ $\qquad (\langle x|p\rangle)$

we have $\langle x|\tilde{p}|\alpha\rangle = -i\hbar \frac{\partial}{\partial x}\langle x|\alpha\rangle$.

Putting $|p\rangle \to |\alpha\rangle$, then $\quad \langle x|\tilde{p}|p\rangle = -i\hbar \frac{\partial}{\partial x}\langle x|p\rangle$.

$\therefore \quad p\langle x|p\rangle = -i\hbar \frac{\partial}{\partial x}\langle x|p\rangle$ ... ~~first~~-order diff. eq..

$\Rightarrow \quad \langle x|p\rangle = N \exp\left[\frac{ipx}{\hbar}\right]$. || N = normalization constant.

To get N, use

$$\delta(x'-x'') = \langle x'|x''\rangle = \int dp\, \langle x'|p\rangle\langle p|x''\rangle$$

$$= |N|^2 \int dp\, \exp\left[\frac{ip(x'-x'')}{\hbar}\right]$$

$$N = \frac{1}{\sqrt{2\pi\hbar}} \quad \longleftarrow \quad = 2\pi\hbar\, |N|^2\, \delta(x'-x'')$$

$$\therefore \quad \boxed{\langle x|p\rangle = \frac{1}{\sqrt{2\pi\hbar}}\, e^{\frac{ipx}{\hbar}}}$$

|| "a" definition of $\delta(x)$

$$\delta(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} dk\, e^{ikx}$$

$\Rightarrow$ Transformations. "It's F.T. and inverse F.T."

- $\langle x|\alpha\rangle = \int dp \, \langle x|p\rangle\langle p|\alpha\rangle \Rightarrow$

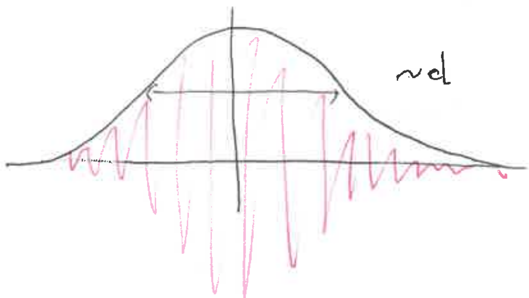
$$\psi_\alpha(x) = \frac{1}{\sqrt{2\pi\hbar}} \int dp \; e^{\frac{ipx}{\hbar}} \, \phi_\alpha(p)$$

- $\langle p|\alpha\rangle = \int dx \, \langle p|x\rangle\langle x|\alpha\rangle \Rightarrow$

$$\phi_\alpha(p) = \frac{1}{\sqrt{2\pi\hbar}} \int dx \; e^{-\frac{ipx}{\hbar}} \, \psi_\alpha(x)$$

(4) Gaussian Wave Packets.

- In Position Space, $\langle x|\alpha\rangle = \left[\frac{1}{\pi^{1/4}\sqrt{d}}\right] \exp\left[ikx - \frac{x^2}{2d^2}\right]$



: "minimum" uncertainty wave packet.

$$\langle \Delta \tilde{x}^2\rangle\langle \Delta \tilde{p}^2\rangle = \frac{\hbar^2}{4}$$ "

- Expectation values :

$$\langle \tilde{x}\rangle = \int_{-\infty}^{\infty} dx \, \langle\alpha|x\rangle x \langle x|\alpha\rangle = 0.$$

$$\langle \tilde{x}^2\rangle = \frac{d^2}{2}, \quad \to \langle(\Delta\tilde{x})^2\rangle = \langle\tilde{x}^2\rangle - \langle\tilde{x}\rangle^2 = \frac{d^2}{2}$$

$$\langle\tilde{p}\rangle = \hbar k, \quad \langle\tilde{p}^2\rangle = \frac{\hbar^2}{2d^2} + \hbar^2 k^2.$$

$$\to \langle(\Delta\tilde{p})^2\rangle = \langle\tilde{p}^2\rangle - \langle\tilde{p}\rangle^2 = \frac{\hbar^2}{2d^2}$$

- momentum-space wave function :

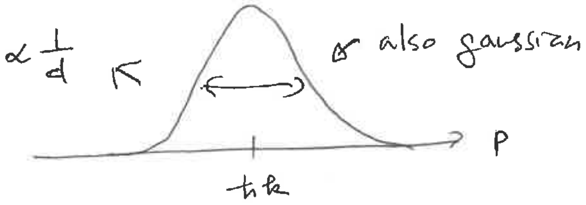
$$\langle p|\alpha\rangle = \frac{1}{\sqrt{2\pi\hbar}} \int_{-\infty}^{\infty} dx \; e^{-\frac{ipx}{\hbar}} \langle x|\alpha\rangle.$$

$$\cancel{= \sqrt{\frac{d}{\hbar\sqrt{\pi}}} \exp \frac{}{\hbar}}$$

$$\langle p|\alpha\rangle = \frac{1}{\sqrt{2\pi\hbar}}\left(\frac{1}{\pi^{1/4}\sqrt{d}}\right)\int_{-\infty}^{\infty} dx \exp\left(-\frac{i}{\hbar}(p-\hbar k)x - \frac{x^2}{2d^2}\right)$$

$$= \sqrt{\frac{d}{\hbar\sqrt{\pi}}}\exp\left[-\frac{d^2}{2\hbar^2}(p-\hbar k)^2\right]$$

$$\int_{-\infty}^{\infty} dx\, e^{-ax^2-bx} = \int_{-\infty}^{\infty} dx\, e^{-a(x+\frac{b}{2a})^2+\frac{b^2}{4a}} = \sqrt{\frac{\pi}{a}}\, e^{\frac{b^2}{4a}}$$



As it gets broader in x-space, it gets sharper in p-space!

ex. $d \to \infty$: infinite uncertainty in x-space

$\to$ δ-function in p-space.

(i.e. momentum is sharply defined!)

$\to$ eigenstate!

(5) Generalization to 3D : trivial!

- $\underset{\sim}{\vec{x}}|\vec{x}\rangle = \vec{x}|x\rangle$, $\underset{\sim}{\vec{p}}|\vec{p}\rangle = \vec{p}|\vec{p}\rangle$

indication of 3D

- $\langle\vec{p}'|\vec{p}''\rangle = \delta^{3}(\vec{p}'-\vec{p}'')$, $\langle\vec{x}'|\vec{x}''\rangle = \delta^{3}(\vec{x}'-\vec{x}'')$

- $\delta^3(\vec{x}'-\vec{x}'') \equiv \delta(x'-x'')\,\delta(y'-y'')\,\delta(z-z'')$

- Completeness : $\int d^3x\, |\vec{x}\rangle\langle\vec{x}| = 1$, $\int d^3p\, |\vec{p}\rangle\langle\vec{p}| = 1$

- $\langle\beta|\underset{\sim}{\vec{p}}|\alpha\rangle = \int d^3x\, \psi_\beta^*(\vec{x})\,(-i\hbar\nabla)\,\psi_\alpha(\vec{x})$

$$\langle\vec{x}|\vec{p}\rangle = \left[\frac{1}{(2\pi\hbar)^{3/2}}\right]\exp\left(\frac{i\vec{p}\cdot\vec{x}}{\hbar}\right)$$

$$\psi_\alpha(x) = \frac{1}{(2\pi\hbar)^{3/2}}\int d^3p\, e^{\frac{i\vec{p}\cdot\vec{x}}{\hbar}}\phi_\alpha(\vec{p}), \quad \phi_\alpha(\vec{p}) = \frac{1}{(2\pi\hbar)^{3/2}}\int d^3x\, e^{-\frac{i\vec{p}\cdot\vec{x}}{\hbar}}\psi_\alpha(\vec{x})$$